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A STUDY OF IDEMPOTENT ELEMENTS IN VARIANTS OF FINITE FULL AND PARTIAL TRANSFORMATION SEMIGROUPS

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ABSTRACT
This study was carried out to synthesize existing result under characterization of idempotent elements in the variant of a finite full transformation semigroup. And as a consequence of the previous results, now the study aim to add to the work done " a study of idempotent elements in a variant of finite partial transformation semigroup" with the goal of determining the behavior, the properties and the relations between the elements of the two semigroups and further address the gaps with the recommendation established earlier. This study reveals novel insights into the Characterization of idempotent elements in the variant of finite partial transformation semigroup. This research contributes to a more profound understanding of the intricacies of these semigroups, laying the groundwork for future explorations in algebraic structures and transformation semigroups.

Keywords: Semigroup, variant Semigroup, transformations, full transformation, Partial transformation & Idempotent element.

1. Introduction

Semigroup is one of the branch of algebra with its own goals, research techniques, & problem formulations. The main reason why semigroups appear in mathematics is because of self-maps of some form that are frequently of interest and whenever f, g and h are such maps, it follows that $(goh) = (fog)h$. The proper abstract concept is that of a group if the mappings are bijections; if not, the study must unavoidably take into account a semigroup. The natural numbers N under addition or multiplication are clear examples of semigroups; a semigroup is just a set S that is closed under an associative binary operation. If the semigroup S satisfies the property that $\forall$ $x, y \in S$ the study have $xy = yx$, then the study say that the semigroup S is a commutative semigroup.

A semigroup S is called a monoid if it contains an identity, otherwise the study can easily adjoin an extra identity 1, or zero to the semigroup S without zero in order to make them monoid. Thus, the study write S^1 and S^0 respectively to denote a semigroup S that adjoined with either 1 or 0.

That is $S^1 = \begin{cases} S & \text{if S has identity element,} \\ S \cup \{1\} & \text{othewise} \end{cases}$ & $S^0 = \begin{cases} S & \text{if S has identity element,} \\ S \cup \{0\} & \text{othewise} \end{cases}$

where $0x = x0 = 0, \ \forall x \in S.$ where $1x = x1 = x, \ \forall x \in S.$

Most researchers first got interested in semigroups as natural extensions of groups and it is easy to see that the roots of group theory lie in the analysis of the actions of a set and the group of permutations of that set.

For instance: Let M be a finite set , say $M = \{m_1, m_2, m_3 \cdots m_n\}$. A transformation of M is an array of the form $\alpha = \begin{pmatrix} m_1 & m_2 & m_3 & \cdots & m_n \\ k_1 & k_2 & k_3 & \cdots & k_n \end{pmatrix}$ where all $k_i \in M$.

Now Consider the finite set X = { 1, 2, 3, 4, 5, 6, 7 } Define $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 5 & 7 & 5 & 2 & 7 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 5 \end{pmatrix}$ & $\delta = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 1 & 6 \end{pmatrix}$. Then each of these mappings α, β and δ shown above represents a transformation (where β and δ are strictly partial transformations).

The main objects of study in this work are finite sets and its transformations. It was noted that the semigroup $Sing_X$ of all singular transformations on a finite set X is generated by its idempotent and this has attracted a lot of interest in the investigation of various types in a semigroup. A more accessible account of the results (transformation) may be found in the monograph of Ganyuskhin & Mazorchuk (2009), "Classical Finite Transformation Semigroups".

2. Related Work

This section presents the essential key definitions, lemmas, & established results that will be crucial in developing theory and proving main assertion to establish a solid framework.

(i) A full transformation is a map from { 1, 2, · · · , n } to { 1, 2, · · · , n } which can be written as $\alpha = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1\alpha & 2\alpha & 3\alpha & \cdots & n\alpha \end{pmatrix}$ and contains n^n elements.

For instance, if $M = \{1, 2,\}$ then the elements of the full transformation denoted by T_2 is as follows: $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

(ii) A partial transformation is a partial map on {1, 2, · · · , n}. If A is a subset of M, then the mapping $\alpha: A \to M$ is called a partial transformation of the finite set M and the set of all partial transformation is denoted by P_n (which contains $(n + 1)^n$ elements).

For instance, if $M = \{1, 2,\}$ then the structure of the partial transformation semigroup of the above set M which denoted by P_2 is as follows:

$\left\{\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, (\quad)\right\}$ *the empty map ( ) is the zero of* P_n.

(iii) An element $e \in S$ where S is a semigroup, is called an idempotent if $e^2 = e$ and the study denote the set of all idempotents in S by $E(S)$.

For instance : Let $e_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ & $e_2 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$ be elements in T_2, then by definition, the study have $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$

Lemma 2.1: Let $\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix} \in S$ be a fixed element with $a_i \in A_i$ and let $\beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_m \\ b_1 & b_2 & \cdots & b_m \end{pmatrix} \in S^\alpha$ be an element of rank $m \leq k$. Then β is an idempotent in S^α iff ∃ a mapping f : {1, · · · , m} → {1, · · · , k}: $b_i \in A_{f(i)}$ & $a_{f(i)} \in B_i$ ∀ i = 1, · · · , m.

3. Materials and Methods

The method of research adopted in this work is to consult the necessary and relevant papers in the literature on variants of transformation semigroup and crucial analysis of existing results related to idempotent generation, therefore the study list some known results, definitions and notations that will be used throughout this work and present some lemmas to prove assertions

Variant of a finite Partial transformations semigroups (P_n^α):

This section focuses on the partial transformation semigroup, denoted by P_n·, consisting of all partial maps from subsets of the set {1, 2, ..., n} to itself. The operation on this semigroup is composition, where two partial maps are combined to form a new partial map. In other words, P_n. comprises all possible mappings that assign elements from {1, 2, ..., n} to its subset, with the possibility of undefined values for some elements. The composition operation allows us to chain these mappings together, creating new transformations.

For example, consider {1, 2, 3}. A partial map α might map 1 to 2 and leave 3 undefined. Another partial map β might map 2 to 3 and leave 1 undefined. The composition αβ would then map 1 to 3 and leave 2 undefined. Generally, if $X = \{1, 2, 3\}$ then the structure of the elements for the finite partial transformation denoted by P_3 is as follows.

$$\left\{\begin{array}{c}
\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix},\begin{pmatrix}1&2&3\\1&1&2\end{pmatrix},\begin{pmatrix}1&2&3\\1&1&3\end{pmatrix},\begin{pmatrix}1&2&3\\1&2&1\end{pmatrix},\begin{pmatrix}1&2&3\\1&2&2\end{pmatrix},\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\1&3&1\end{pmatrix},\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix},\begin{pmatrix}1&2&3\\1&3&3\end{pmatrix},\\
\begin{pmatrix}1&2&3\\2&1&1\end{pmatrix},\begin{pmatrix}1&2&3\\2&1&2\end{pmatrix},\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix},\begin{pmatrix}1&2&3\\2&2&1\end{pmatrix},\begin{pmatrix}1&2&3\\2&2&2\end{pmatrix},\begin{pmatrix}1&2&3\\2&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix},\begin{pmatrix}1&2&3\\2&3&2\end{pmatrix},\begin{pmatrix}1&2&3\\2&3&3\end{pmatrix},\\
\begin{pmatrix}1&2&3\\3&1&1\end{pmatrix},\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix},\begin{pmatrix}1&2&3\\3&1&3\end{pmatrix},\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix},\begin{pmatrix}1&2&3\\3&2&2\end{pmatrix},\begin{pmatrix}1&2&3\\3&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\3&3&1\end{pmatrix},\begin{pmatrix}1&2&3\\3&3&2\end{pmatrix},\begin{pmatrix}1&2&3\\3&3&3\end{pmatrix}\\
\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}2\\2\end{pmatrix},\begin{pmatrix}2\\3\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}3\\3\end{pmatrix}\\
\begin{pmatrix}1&2\\1&1\end{pmatrix},\begin{pmatrix}1&2\\1&2\end{pmatrix},\begin{pmatrix}1&2\\1&3\end{pmatrix},\begin{pmatrix}1&2\\2&1\end{pmatrix},\begin{pmatrix}1&2\\2&2\end{pmatrix},\begin{pmatrix}1&2\\2&3\end{pmatrix},\begin{pmatrix}1&2\\3&1\end{pmatrix},\begin{pmatrix}1&2\\3&2\end{pmatrix},\begin{pmatrix}1&2\\3&3\end{pmatrix}\\
\begin{pmatrix}1&3\\1&1\end{pmatrix},\begin{pmatrix}1&3\\1&2\end{pmatrix},\begin{pmatrix}1&3\\1&3\end{pmatrix},\begin{pmatrix}1&3\\2&1\end{pmatrix},\begin{pmatrix}1&3\\2&2\end{pmatrix},\begin{pmatrix}1&3\\2&3\end{pmatrix},\begin{pmatrix}1&3\\3&1\end{pmatrix},\begin{pmatrix}1&3\\3&2\end{pmatrix},\begin{pmatrix}1&3\\3&3\end{pmatrix}\\
\begin{pmatrix}2&3\\1&1\end{pmatrix},\begin{pmatrix}2&3\\1&2\end{pmatrix},\begin{pmatrix}2&3\\1&3\end{pmatrix},\begin{pmatrix}2&3\\2&1\end{pmatrix},\begin{pmatrix}2&3\\2&2\end{pmatrix},\begin{pmatrix}2&3\\2&3\end{pmatrix},\begin{pmatrix}2&3\\3&1\end{pmatrix},\begin{pmatrix}2&3\\3&2\end{pmatrix},\begin{pmatrix}2&3\\3&3\end{pmatrix}\\
\begin{pmatrix}\ \end{pmatrix}
\end{array}\right\}$$

Now, the study shift attention to the main object of investigations. That is (P_n^{α}) for $\alpha \in P_n$, a variant of the partial transformation semigroup. This variant is denoted by P_n. Specifically, P_n. consists of all partial maps from subsets of {1, 2, ..., n} to itself, with a redefined operation that differs from the standard composition.

This variant allows us to explore new algebraic structures and relationships, extending understanding of transformation semigroups. This variant offers a unique perspective on transformation semigroups, enabling us to explore new research directions." semigroup.

Definition 3.1: Let S be a semigroup, for a fixed element $\alpha \in S$, the study define a new operation $\star_\alpha$ on S by $x \star_\alpha y = x \alpha y, \ \forall\ x, y \in S$. Then $(S, \star_\alpha)$ is a semigroup called the variant of S with respect to α, and is denoted by S^α.

Example 3.2: Suppose $(N, +)$ is a semigroup denoted by S and pick the element $4 \in N$, then $(N, \star_4)$ is the variant of S with respect to the fixed element 4, where $x \star_4 y = x + 4 + y$ for all $x, y \in S$ and is denoted by S^4. This *N* together with the new binary operation $\star_4$ is a semigroup called the variant of S with respect to the fixed element 4, which will be denoted by

S^4. That is, suppose the study take $5 \ \& \ 8 \in (N, +)$, then by the above example, the study can vividly see that $5 \star_4 8 = 5 + 4 + 8 = 17.$

Further, the study the variant of a finite partial transformation semigroup which is denoted by P_n^α and then explore the structure of it using the above definition.

Example 3.3: (i) Let $x = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}$ & $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ be elements in P_3, then by the definition : $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \star_\alpha \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}$

(ii) Let $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $y = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, then by the above definition the study have

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \star_\alpha \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(iii) Let $x = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ and $\alpha = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$, then by the above definition the study have

$$\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \star_\alpha \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

So the study now give the structure for the variant of a finite partial transformation semigroup of the set $X = \{1, 2, \cdots n\}$. which denoted by P_n^α for the element $\alpha = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 1\alpha & 2\alpha & 3\alpha & \ldots & n \end{pmatrix}$,
To examine the structure of the variant of a finite partial transformation semigroup, denoted by P_n^α , the study provide a concise summary of its key details in the following remark."

Remark 3.4 : The elements of the variant of a finite partial transformation semigroup (P_n^α) contains the same as that of the original finite partial transformation (P_n). the study now go on to discuss the following results after explaining the primary focus of this study, which are idempotents of a variant of the finite partial transformation semigroup and listening to some pertinent results about them:

Definition 3.5 [1]: Let S be a semigroup, for any fixed element $\alpha \in S$, the study define a new operation $\star_\alpha$ on S by $x \star_\alpha y = x\,\alpha\,y, \ \forall\ x, y \in S.$ Then $(S, \star_\alpha) \Rightarrow S^\alpha = \{x : x\,\alpha\,y = x\}.$

For instance: Let $x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 1 & 3 & 5 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 4 & 3 & 2 \end{pmatrix}$ & $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 4 & 3 \end{pmatrix}$ be a fixed element, then by the above definition the study have:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 1 & 3 & 5 \end{pmatrix} *_\alpha \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 4 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 1 & 3 & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 1 & 3 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 4 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 4 & 3 & 2 \end{pmatrix}$$

4. RESULTS AND DISCUSSION

Characterisation of idempotent elements in a variant of the finite partial transformations semigroups ($E_\alpha(\mathrm{P}_n^\alpha)$).

Building on the foundational research on idempotent elements in the variants of finite full transformation semigroups in [1], this study aims to extend and generalize those findings to characterize idempotent elements in variants of finite partial transformation semigroups. The study seek to explore the relationships between these idempotent elements and their counterparts in full transformation semigroups. To achieve this, research methodology involves:

1. Conducting a comprehensive literature review of relevant papers on P_n^α
2. Analyzing analogous results on idempotent generation, particularly in T_n.
3. Synthesizing this background information to develop a theoretical framework for characterizing idempotent elements of P_n^α.

By bridging the knowledge gap between full & partial transformations, this research contributes to a deeper understanding of idempotents' properties & behavior in these algebraic structures.

Main Results

This research addresses a critical gap in the literature by extending idempotent element characterization to partial transformation semigroups, methodology ensures a comprehensive understanding of the underlying algebraic structures to present the result of finding.

Definition 4.1: An element $\beta \in S^\alpha$ is called an idempotent if it satisfies the condition $\beta^2 = \beta\alpha\beta = \beta$ and the study denote the set of all idempotents in S^α by $E(S^\alpha)$. $E(S^\alpha) = \{\beta \in S \mid \beta\alpha\beta = \beta\}$.

Definition 4.2: The cardinality of the image of a transformation α (i.e $|\mathrm{im}(\alpha)|$) is called the rank of the transformation and is denoted by $\mathrm{rank}(\alpha)$. And the defect of α can also be defined as $\mathrm{def}(\alpha) = n - \mathrm{rank}(\alpha)$.

Now these two lemmas below serve as crucial stepping stones in investigation, culminating in the primary outcome of this research, the characterization of idempotent elements in P_n^α, but the first (Lemma 4.3:) is an unproved lemma in the monograph of Ganyushkin & Mazorchuk (i.e lemma 13.3.1). So the study proved it to coincide with the result of this work.

Lemma 4.3: Let $\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix} \in T_n$ be a sandwich element with $a_i \in A_i$ and let $\beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_m \\ b_1 & b_2 & \cdots & b_m \end{pmatrix} \in T_n^\alpha$ be an element of rank $m \leq k$. Then β is an idempotent in T_n^α if and only if there exists a mapping $f : \{1, \cdots, m\} \to \{1, \cdots, k\}$ such that $b_i \in A_{f(i)}$ and $a_{f(i)} \in B_i$ for all $i = 1, \cdots, m$.

Proof: To determine such β we shall choose an image for each a_i in the corresponding block of A_i which can be done in different ways, this determines $\mathrm{im}(\beta)$ and after that the study shall arbitrarily choose the image of all elements from $N/im(a)$ inside $\mathrm{dom}(\beta)$, since all choices are independent, the statement now follows by direct computation.

Example 4.3.1: Let $\alpha = \begin{pmatrix} \{12\} & 3 \\ 1 & 3 \end{pmatrix} \in P_3$ be a sandwich element, then an element $\beta = \begin{pmatrix} \{12\} & 3 \\ 2 & 3 \end{pmatrix}$ in P_3^α is an idempotent for the variant P_3^α, by applying the above lemma (4.3.)

Example 4.3.2: Let $\alpha = \begin{pmatrix} 1 & \{24\} & 3 \\ 1 & 4 & 3 \end{pmatrix} \in T_4$ be a sandwich element, Then an element $\beta = \begin{pmatrix} \{13\} & \{24\} \\ 3 & 2 \end{pmatrix}$ in T_4^α is idempotent for the variant T_4^α by applying the above lemma (4.3).

Lemma 4.4: Suppose $\alpha \in P_n$ and $\beta \in P_n^\alpha$. Then the element β is an idempotent of P_n^α if and only if $im(\beta) = \mathrm{fix}(\alpha\beta)$

Proof: Here the study want to demonstrate that a mapping $\beta \in P_n^\alpha$ is an idempotent function. Now, suppose $y \in im(\beta)$, then $\forall\, x \in x:\; y(\alpha\beta) = y$.
Given $\alpha\beta = y$: $x(\beta\alpha\beta) = x\beta(\alpha\beta) = y(\alpha\beta) = y = x\beta, \Rightarrow \beta\alpha\beta = \beta. \therefore \beta$ is idempotent.

These results collectively enable us to:

1. Establish necessary and sufficient conditions for idempotency in P_n^α.
2. Provide a comprehensive characterization of idempotent elements in these semigroups.
3. Shed light on the algebraic properties and behavior of these idempotent elements.

In essence, the two lemmas above form the backbone of characterization, paving the way for a deeper understanding of idempotent elements in variants of finite partial transformation semigroups. So the above lemma 4.3 characterizes & exhibits the idempotent element in P_n^α where "α" the sandwich element is an idempotent. And lemma 4.4: characterizes and exhibits the idempotent elements in P_n^α for "α"not an idempotent, but for an arbitrary element chosen, below are few examples to shed more light.

Example (4.4.1): Let $x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 1 & 3 & 5 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 4 & 3 & 2 \end{pmatrix}$ and $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 4 & 3 & 5 \end{pmatrix}$ be a fixed element, then by the above definition the study have:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 1 & 3 & 5 \end{pmatrix} \star_\alpha \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 4 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 1 & 3 & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 1 & 3 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 4 & 3 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 4 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 1 & 3 & 5 \end{pmatrix}$$

Example (4.4.2): Let $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $y = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\alpha = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, then by the above definition the study have : $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \star_\alpha \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Example (4.4.3): Let $x = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ and $\alpha = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$, then by the above definition the study have: $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \star_\alpha \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$.

In general, based on Definition 4.1 and Lemma 4.4, complemented by illustrative examples, the study have systematically developed a methodology for characterizing idempotent elements in the two key algebraic structures, that is: (1). T_n^α and (2) P_n^α. This comprehensive approach constitutes the core outcome of research. This result offer a significant contribution to the study of transformation semigroups, shedding light on the algebraic characteristics on the idempotents.

Note: In the two results (Lemma 4.3: & Lemma 4.4:) above, it is sufficiently showed that the idempotent of the variant of the finite transformation semigroups are always $\alpha - \text{dependent}$. (i.e an element β may not be idempotent for some α).

Example (4.4.4): Consider the semigroup P_3^α with a sandwich element $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}$. Then the element $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}$ in P_3^α is an idempotent in P_3^α, since the im$\beta = (2,3)$ fixes the product of $\alpha\beta$, that is $\alpha\beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}$. Similarly,
$(\beta\alpha\beta) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}$ is same as the definition of idempotent.

Example (4.4.5): Consider the semigroup P_3^α with a fixed element $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}$. Then the element $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$ in P_3^α is not an idempotent for the above α since the image of $\beta = (1,3)$ does not fixed the product of $\alpha\beta$, i.e $\alpha\beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{pmatrix}$ therefore $(\beta\alpha\beta) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$

However, for another fixed element $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. Then $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$ in T_3^α is an idempotent, since the im$\beta = (1,3)$ fixed the product of $\alpha\beta$, .Thus the element $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$ is not an idempotent for $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}$, since it doesn't satisfy $(\beta\alpha\beta) = \beta$.

Note: The elements which satisfied the condition of idempotent for every fixed elements are those elements with constant map, but it only holds for T_n^α not for all elements of P_n^α (i.e mapping with a rank such that |imβ| = { 1,2,$\cdot\cdot\cdot$n }. Below is an axiom to generalize the details of the statement.

Axiom 4.5: Suppose α is a sandwich element and let $\beta \in P_n^\alpha$. Then the element β is an idempotent of T_n^α (since $T_n^\alpha \subset P_n^\alpha$) for every sandwich element chosen arbitrarily if the range of β is equal to a constant map. That is $\text{im}(\beta) = 1,2,3,\cdot\cdot\cdot,n.\ \forall\, \alpha$ chosen arbitrarily.

Example 4.5.1: Consider the semigroup T_3^α with a sandwich element $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. Then an element $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$ in T_3^α is an idempotent in T_3^α, since the im $\beta = (1)$ which also fixed itself in the product of $\alpha\beta$, that is $\alpha\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$.

Analogously $(\beta\alpha\beta) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$ same as the above definition 4.2

Example 4.5.2: Again for another different value of β, the element $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}$ in T_3^α is also an idempotent in T_3^α, because the im $\beta = (\ 2\ )$ is equal to a constant map as required and similarly the condition $(\beta\alpha\beta) = \beta$ for every α arbitrarily holds: Thus the element β is an idempotent for all fixed element α.

Example 4.5.3: Consider the semigroup T_3^α with element $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and Let $\beta = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$, then it contradicts the axiom, $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \star_\alpha \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = (\quad)$

Remark: So this example 4.5.3**:** suffices to prove the condition that says " idempotent for every sandwich elements are those elements with constant map holds for only T_n^α not ∀ elements of P_n^α ". And the above examples (4.5.1: & 4.5.2:) shows that the idempotents in T_n^α for every sandwich element are those elements with constant maps only (i.e. $\beta \in E(T_n^\alpha)$ is such that $|Im\beta|$ = (1, 2, 3, · · · , n) because they are the only elements that satisfy the condition of Axiom 4.5. and the study called them idempotent of rank(1).

So with this foundation, the study can now outline the methodological framework for characterizing idempotent elements in both full and partial transformations on the set." $X = \{1, 2, 3 \cdots n\}$ ∀ $\alpha = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1\alpha & 2\alpha & 3\alpha & \cdots & n\alpha \end{pmatrix}$ in the table below, while noting that the rest of the elements are $\alpha -$ dependent.

Table 4.5.4:

Section	Description	Condition	Examples	Idempotents
Definition 4.1	Idempotent element	$\beta^2 = \beta$	________	________
Lemma 4.3	Sandwich element	$\exists\ f : \{m\} \to \{k\}\ \forall\ i\ \in m$	4.3.1	Idempotent
Lemma 4.4	characterizes idempotents	$im(\beta) = \text{fix}(\alpha\beta)$	4.4.1	Idempotent
Axiom 4.5	constant map element	$im(\beta)$ = constant map	4.5.1	Idempotent
Example 4.5.3	Fixed element	$im(\beta)$ = constant map	4.5.3	Not idempotent

. . . . .	. . . . .	. . . . .	. . .	. . . . .
. . . . .	. . . . .	. . . . .	. . .	. . . . .
. . . . .	. . . . .	. . . . .	. . .	. . . . .

5. CONCLUSION

This study successfully elucidated the main structure and outcomes of idempotent elements within both finite full and partial transformation semigroups, expanding on previous research [1]. comprehensive analysis revealed that the elements in the variant semigroups are identical to those in the original semigroups, providing a foundational understanding of their characteristics. Moreover, investigation uncovered a crucial distinction between the original and variant of the finite partial transformation semigroups. Specifically, the study found that the condition for characterizing the idempotents in the variant of finite partial transformation semigroup does not hold for certain elements [Example 4.5.3:], particularly in the subset elements. This discovery underscores the significance of considering the unique properties of each variant semigroup, as they may exhibit distinct behaviors and characteristics. The research provides a thorough examination of idempotent elements in finite full and partial transformation semigroups, shedding light on their characteristics and behaviors. The distinctions uncovered between the original and variant semigroups emphasize the importance of careful consideration and analysis.

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