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ORDER AND STABILITY OF THE REFORMULATED HYBRID BLOCK METHOD FOR SOLVING NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, the order and stability analysis of the hybrid variable step size block method for solving nonlinear ordinary differential equation is established. The scheme adopted a variable step size technique. The new hybrids block method for solving stiff ODEs developed by Sagir et. al., are re-derived by introducing a variable step size ratio (r), $r = \frac{1}{5}$ to obtained a stable method. However, the paper focuses on analysing the order and stability of the new stable method, by establishing the necessary and sufficient condition for order and stability. The method is found to satisfy the entire criteria for order and Stability, so the scheme is consistent, Zero stable and A-Stable capable of solving stiff nonlinear initial value problems. Existing Stiff Nonlinear Ordinary Differential Equations is solved using the method and the numerical result obtained using the new method are found to be efficient at certain step sizes. Hence, the new scheme is recommended for the solution of stiff Nonlinear Ordinary Differential Equations.

Keywords: Hybrid, Block Method, Stiff, Order, zero stable, A-Stable

Introduction

A fascinating feature of mathematics is its capacity to convert real-world problems into mathematical models and then solve using appropriate methods. Most scientific and engineering problems are express in the form of differential equations. Such equations can either be Linear or Nonlinear differential equations. The ultimate aim of researchers who formulate these equation is to find their solution. Unfortunately, often times the modeled equations turn to be stiff in nature. Consider a system of first order initial value problems (IVPs) of the form:

$$y' = f(x, y) \qquad X \in [a, b] \qquad y(X_0) = y_0 \tag{1}$$

System (1) can be regarded as stiff if its exact solution contains very fast as well as very slow component (Dahquist, 1974). Stiff initial value problems arise in numerous fields of engineering and the physical science. They are particularly found in electrical circuits, vibrations, chemical reactions, kinetics, authomated control, combustion, theory of fluid mechanics e.t.c. The solution is characterized by the presence of transient and steady state component which restrict the step size of many numerical method except method with A-stable properties,this can be found in (Suleiman *et al.,* 2013).

Stiff problem usually deviates from been solved analytically due to its complexities and other phenomena which is found within its solution, the transient and steady state components found in its solution make explicit method complex to handle with a very good results. While, numerical solution is much more easier and obtainable in any form of stiff Initial Value Problems of ODEs. Most of the stiff cases have no analytical solutions at all. Hence, choices are always made to what numerical methods would solve any sort of stiff Initial Value Problems of ODEs. The final achievement is to get a method with a solution that has absolutely minimum scale error and computational time. Backward differentiation formula came to be developed by (Curtiss and Hirschfield, 1952), several extensions of the method was carrry out by many scholars incuding (Cash, 1980, 2000) and the block aspect was formulated by (Ibrahim *et al.,* 2007). Different scholars work tremendeously in making a BBDF method that can handle stiff IVPs with minimum error and computational time, these can be found in (Musa *et al.,* 2014, 2012, 2014), (Sagir, 2014, 2013, 2012), (Abdullahi *et al.,* 2022), (Abdullah and Musa, 2021, 2019), (Nasarudden *et al.,* 2020), (Sagir and Abdullahi, 2022), (Sagir *et al.,* 2023) and (Ibrahim *et al.,* 2025).

Definition 1.1: A general k-step linear multistep method is defined as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{2}$$

Where α_j and β_j are constant and $\alpha_k \neq 0$. α_0 and β_0 cannot both be zero at the same time. For any k step method, α_k is normalized to one. The method is said to be explicit if $\beta_k = 0$ and implicit if $\beta_k \neq 0$.

Definition 1.2: The first and second characteristics polynomials of the equation (2) are define by

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j \tag{3}$$

And

$$\sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^j \tag{4}$$

Definition 1.3: Let Y_m and F_m be vectors defined by

$$Y_m = [y_n, y_{n+1}, y_{n+2}, \dots, y_{n+r-1}]^{\mathrm{T}},$$

$$F_m = [f_n, f_{n+1}, f_{n+2}, \dots f_{n+r-1}]^{\mathrm{T}} \text{ respectively}$$

Then a general k-block, r-point method is a matrix finite difference equation of the form

$$Y_m = \sum_{i=1}^{k} A_i \, Y_{m-1} + \mathrm{h} \sum_{i=0}^{k} B_i \, f_{m-1} \qquad (5)$$

Where all A_i's and B_i's are properly chosen r × r matrix coefficients and m = 0,1,2,... represent the block number, n = mr the first step number in the m-th block and r is the proposed block size.

Definition 1.4: The block method (4) is said to be zero stable if the root $R_{(j,j)} = 1(1)$ k of the first characteristics polynomial $\rho(R)$= det $[\sum_{i=0}^{k} A_i \, R^{k-i}] = 0$, $A_o = -1$, satisfies $|R_j| < 1$. If one of the roots is +1, we call this root the principal root of $\rho(R)$.

This study consider the block backward differentiation aspect in (Sagir *et al.,* 2022) and introduced a variable r. It is of the form:

$$\sum_{j=0}^{2} \alpha_{j,i,r} y_{n+j-2} + \sum_{j=0}^{3} \alpha_{j+3,i} y_{n+(j+1/2)} = h\beta_{k+1i} f_{n+k} \qquad k = \frac{1}{2}, 1, \frac{3}{2}, 2 \qquad (6)$$

The method approximates two solution values of y_{n+1}, y_{n+2} and two off-step values of $y_{n+\frac{1}{2}}$, $y_{n+\frac{3}{2}}$ at a time simultaneously, per integration step for a stiff ODE.

Methodology

1 Formulation of the Method

In this section, two approximate solution values y_{n+1} and y_{n+2} with step size h, and two off-step points $y_{n+\frac{1}{2}}$ and $y_{n+\frac{3}{2}}$ which are chosen at the points where the step size is halved are formulated in a block simultaneously. The formulae are computed using two back values y_n and y_{n-1} with step size h.

The method (6) is used in this formulation, where k *and* i have the same value. The formula (6) is derived using Taylor's series expansion about x_n

Definition: According to [17], the linear operator L_i associated with first, second, third and fourth point of the method with off-step point method is defined as follows:

Consider the following value of k & $i's$ value for the cases below:

$$L_i[y(x_n), h]: \alpha_{0,i} y_{n-2} + \alpha_{1,i} y_{n-1} + \alpha_{2,i} y_n + \alpha_{3,i} y_{n+\frac{1}{2}} + \alpha_{4,i} y_{n+1} + \alpha_{5,i} y_{n+\frac{3}{2}} + \alpha_{6,i} y_{n+2} - h\beta_{k+1,i} f_{n+k} = 0 \qquad (7)$$

CASE 1: $k = i = \frac{1}{2}$

$$L_{\frac{1}{2}}[y(x_n), h]: \alpha_{0,\frac{1}{2}} y_{n-2} + \alpha_{1,\frac{1}{2}} y_{n-1} + \alpha_{2,\frac{1}{2}} y_n + \alpha_{3,\frac{1}{2}} y_{n+\frac{1}{2}} + \alpha_{4,\frac{1}{2}} y_{n+1} + \alpha_{5,\frac{1}{2}} y_{n+\frac{3}{2}} + \alpha_{6,\frac{1}{2}} y_{n+2} - h\beta_{\frac{3}{2},\frac{1}{2}} \left[f\left(x_n + \frac{1}{2} h \right) \right] \qquad (8)$$

CASE 2: $k = i = 1$

$$L_1[y(x_n),h]\colon \alpha_{0,1}y_{n-2}+\alpha_{1,1}y_{n-1}+\alpha_{2,1}y_n+\alpha_{3,1}y_{n+\frac{1}{2}}+\alpha_{4,1}y_{n+1}+\alpha_{5,1}y_{n+\frac{3}{2}}+\alpha_{6,1}y_{n+2} - h\beta_{2,1}[f(x_n+h)] = 0 \quad (9)$$

CASE 3: $k = i = \frac{3}{2}$

$$L_{\frac{3}{2}}[y(x_n),h]\colon \alpha_{0,\frac{3}{2}}y_{n-2}+\alpha_{1,\frac{3}{2}}y_{n-1}+\alpha_{2,\frac{3}{2}}y_n+\alpha_{3,\frac{3}{2}}y_{n+\frac{1}{2}}+\alpha_{4,\frac{3}{2}}y_{n+1}+\alpha_{5,\frac{3}{2}}y_{n+\frac{3}{2}}+\alpha_{6,\frac{3}{2}}y_{n+2} - h\beta_{\frac{5}{2},\frac{3}{2}}\left[f\left(x_n+\frac{3}{2}h\right)\right] = 0 \qquad (10)$$

CASE 4: $k = i = 2$

$$L_2[y(x_n),h]\colon \alpha_{0,2}y_{n-2}+\alpha_{1,2}y_{n-1}+\alpha_{2,2}y_n+\alpha_{3,2}y_{n+\frac{1}{2}}+\alpha_{4,2}y_{n+1}+\alpha_{5,2}y_{n+\frac{3}{2}}+\alpha_{6,2}y_{n+2} - h\beta_{3,2}[f(x_n+2h)] = 0 \qquad (11)$$

From cases 8, 9, 10 & 11, we have the following linear operators

$$\left.\begin{array}{r}
L_{\frac{1}{2}}[y(x_n),h]\colon \alpha_{0,\frac{1}{2}}y_{n-2}+\alpha_{1,\frac{1}{2}}y_{n-1}+\alpha_{2,\frac{1}{2}}y_n+\alpha_{3,\frac{1}{2}}y_{n+\frac{1}{2}}+\alpha_{4,\frac{1}{2}}y_{n+1}+\alpha_{5,\frac{1}{2}}y_{n+\frac{3}{2}}+\alpha_{6,\frac{1}{2}}y_{n+2} - h\beta_{\frac{3}{2},\frac{1}{2}}\left[f\left(x_n+\frac{1}{2}h\right)\right]=0\\
L_1[y(x_n),h]\colon \alpha_{0,1}y_{n-2}+\alpha_{1,1}y_{n-1}+\alpha_{2,1}y_n+\alpha_{3,1}y_{n+\frac{1}{2}}+\alpha_{4,1}y_{n+1}+\alpha_{5,1}y_{n+\frac{3}{2}}+\alpha_{6,1}y_{n+2} - h\beta_{2,1}[f(x_n+h)]=0\\
L_{\frac{3}{2}}[y(x_n),h]\colon \alpha_{0,\frac{3}{2}}y_{n-2}+\alpha_{1,\frac{3}{2}}y_{n-1}+\alpha_{2,\frac{3}{2}}y_n+\alpha_{3,\frac{3}{2}}y_{n+\frac{1}{2}}+\alpha_{4,\frac{3}{2}}y_{n+1}+\alpha_{5,\frac{3}{2}}y_{n+\frac{3}{2}}+\alpha_{6,\frac{3}{2}}y_{n+2} - h\beta_{\frac{5}{2},\frac{3}{2}}\left[f\left(x_n+\frac{3}{2}h\right)\right]=0\\
L_2[y(x_n),h]\colon \alpha_{0,2}y_{n-2}+\alpha_{1,2}y_{n-1}+\alpha_{2,2}y_n+\alpha_{3,2}y_{n+\frac{1}{2}}+\alpha_{4,2}y_{n+1}+\alpha_{5,2}y_{n+\frac{3}{2}}+\alpha_{6,2}y_{n+2} - h\beta_{3,2}[f(x_n+2h)]=0
\end{array}\right\}$$

Expanding (x_n-2h),(x_n-h), $y(x_n)$,$y(x_n+\frac{1}{2}h)$,$y(x_n+h)$,$y(x_n+\frac{3}{2}h)$,$y(x_n+2h)$,$f\left(x_n+\frac{1}{2}h\right)$,$f\left(x_n+\frac{3}{2}h\right)$,$f(x_n+h)$,$f(x_n+2h)$ in (8), (9), (10) and (11) with a Taylor's series expansion about x_n and collect the like terms and rearrange, we have the following

$$\left.\begin{array}{r}
C_{0,\frac{1}{2}} = \alpha_{0,\frac{1}{2}}+\alpha_{1,\frac{1}{2}}+\alpha_{2,\frac{1}{2}}+\alpha_{4,\frac{1}{2}}+\alpha_{5,\frac{1}{2}}+\alpha_{6,\frac{1}{2}} = -1\\
C_{1,\frac{1}{2}} = -2r\alpha_{0,\frac{1}{2}} - r\alpha_{1,\frac{1}{2}} + \alpha_{4,\frac{1}{2}} + \frac{3}{2}\alpha_{5,\frac{1}{2}} + 2\alpha_{6,\frac{1}{2}} - \beta_{\frac{3}{2},\frac{1}{2}} = -\frac{1}{2}\\
C_{2,\frac{1}{2}} = 2r^2\alpha_{0,\frac{1}{2}} + \frac{1}{2}r^2\alpha_{1,\frac{1}{2}} + \frac{1}{2}\alpha_{4,\frac{1}{2}} + \frac{9}{8}\alpha_{5,\frac{1}{2}} + 2\alpha_{6,\frac{1}{2}} - \frac{1}{2}\beta_{\frac{3}{2},\frac{1}{2}} = -\frac{1}{8}\\
C_{3,\frac{1}{2}} = -\frac{4}{3}r^3\alpha_{0,\frac{1}{2}} - \frac{1}{6}r^3\alpha_{1,\frac{1}{2}} + \frac{1}{6}\alpha_{4,\frac{1}{2}} + \frac{27}{48}\alpha_{5,2} + \frac{4}{3}\alpha_{6,\frac{1}{2}} - \frac{1}{8}\beta_{\frac{3}{2},\frac{1}{2}} = -\frac{1}{48}\\
C_{4,\frac{1}{2}} = \frac{2}{3}r^4\alpha_{0,\frac{1}{2}} + \frac{1}{24}r^4\alpha_{1,\frac{1}{2}} + \frac{1}{24}\alpha_{4,\frac{1}{2}} + \frac{81}{384}\alpha_{5,\frac{1}{2}} + \frac{2}{3}\alpha_{6,\frac{1}{2}} - \frac{1}{48}\beta_{\frac{3}{2},\frac{1}{2}} = -\frac{1}{384}\\
C_{5,\frac{1}{2}} = -\frac{4}{15}r^5\alpha_{0,\frac{1}{2}} - \frac{1}{120}r^5\alpha_{1,\frac{1}{2}} + \frac{1}{120}\alpha_{4,\frac{1}{2}} + \frac{243}{3840}\alpha_{5,\frac{1}{2}} + \frac{4}{15}\alpha_{6,\frac{1}{2}} - \frac{1}{384}\beta_{\frac{3}{2},\frac{1}{2}} = -\frac{1}{3840}\\
C_{6,\frac{1}{2}} = \frac{4}{45}r^6\alpha_{0,\frac{1}{2}} + \frac{1}{720}r^6\alpha_{1,\frac{1}{2}} + \frac{1}{720}\alpha_{4,\frac{1}{2}} + \frac{729}{46080}\alpha_{5,\frac{1}{2}} + \frac{4}{45}\alpha_{6,\frac{1}{2}} - \frac{1}{3840}\beta_{\frac{3}{2},\frac{1}{2}} = -\frac{1}{46080}
\end{array}\right\} \quad (12)$$

$$\left.\begin{aligned}
C_{0,1} &= \alpha_{0,1}+\alpha_{1,1}+\alpha_{2,1}+\alpha_{4,1}+\alpha_{5,1}+\alpha_{6,1}=-1\\
C_{1,1} &= -2r\alpha_{0,1}-r\alpha_{1,1}+\frac{1}{2}\alpha_{4,1}+\frac{3}{2}\alpha_{5,1}+2\alpha_{6,1}-\beta_{2,1}=-1\\
C_{2,1} &= 2r^2\alpha_{0,1}+\frac{1}{2}r^2\alpha_{1,1}+\frac{1}{8}\alpha_{4,1}+\frac{9}{8}\alpha_{5,1}+2\alpha_{6,1}-2\beta_{2,1}=-\frac{1}{2}\\
C_{3,1} &= -\frac{4}{3}r^3\alpha_{0,1}-\frac{1}{6}r^3\alpha_{1,1}+\frac{1}{48}\alpha_{4,1}+\frac{27}{48}\alpha_{5,2}+\frac{4}{3}\alpha_{6,1}-2\beta_{2,1}=-\frac{1}{6}\\
C_{4,1} &= \frac{2}{3}r^4\alpha_{0,1}+\frac{1}{24}r^4\alpha_{1,1}+\frac{1}{384}\alpha_{4,1}+\frac{81}{384}\alpha_{5,1}+\frac{2}{3}\alpha_{6,1}-\frac{4}{3}\beta_{2,1}=-\frac{1}{24}\\
C_{5,1} &= -\frac{4}{15}r^5\alpha_{0,1}-\frac{1}{120}r^5\alpha_{1,1}+\frac{1}{3840}\alpha_{4,1}+\frac{243}{3840}\alpha_{5,1}+\frac{4}{15}\alpha_{6,1}-\frac{2}{3}\beta_{2,1}=-\frac{1}{120}\\
C_{6,1} &= \frac{4}{45}r^6\alpha_{0,1}+\frac{1}{720}r^6\alpha_{1,1}+\frac{1}{46080}\alpha_{4,1}+\frac{729}{46080}\alpha_{5,1}+\frac{4}{45}\alpha_{6,1}-\frac{4}{15}\beta_{2,1}=-\frac{1}{720}
\end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned}
C_{0,\frac{3}{2}} &= \alpha_{0,\frac{3}{2}}+\alpha_{1,\frac{3}{2}}+\alpha_{2,\frac{3}{2}}+\alpha_{4,\frac{3}{2}}+\alpha_{5,\frac{3}{2}}+\alpha_{6,\frac{3}{2}}=-1\\
C_{1,\frac{3}{2}} &= -2r\alpha_{0,\frac{3}{2}}-r\alpha_{1,\frac{3}{2}}+\frac{1}{2}\alpha_{4,\frac{3}{2}}+\alpha_{5,\frac{3}{2}}+2\alpha_{6,\frac{3}{2}}-\beta_{\frac{5}{2},\frac{3}{2}}=-\frac{3}{2}\\
C_{2,\frac{1}{2}} &= 2r^2\alpha_{0,\frac{1}{2}}+\frac{1}{2}r^2\alpha_{1,\frac{1}{2}}+\frac{1}{8}\alpha_{4,\frac{1}{2}}+\frac{1}{2}\alpha_{5,\frac{1}{2}}+2\alpha_{6,\frac{1}{2}}-\frac{3}{2}\beta_{\frac{5}{2},\frac{3}{2}}=-\frac{9}{8}\\
C_{3,\frac{1}{2}} &= -\frac{4}{3}r^3\alpha_{0,\frac{1}{2}}-\frac{1}{6}r^3\alpha_{1,\frac{1}{2}}+\frac{1}{48}\alpha_{4,\frac{1}{2}}+\frac{1}{6}\alpha_{5,2}+\frac{4}{3}\alpha_{6,\frac{1}{2}}-\frac{9}{8}\beta_{\frac{5}{2},\frac{3}{2}}=-\frac{27}{48}\\
C_{4,\frac{1}{2}} &= \frac{2}{3}r^4\alpha_{0,\frac{1}{2}}+\frac{1}{24}r^4\alpha_{1,\frac{1}{2}}+\frac{1}{384}\alpha_{4,\frac{1}{2}}+\frac{1}{24}\alpha_{5,\frac{1}{2}}+\frac{2}{3}\alpha_{6,\frac{1}{2}}-\frac{27}{48}\beta_{\frac{5}{2},\frac{3}{2}}=-\frac{81}{384}\\
C_{5,\frac{1}{2}} &= -\frac{4}{15}r^5\alpha_{0,\frac{1}{2}}-\frac{1}{120}r^5\alpha_{1,\frac{1}{2}}+\frac{1}{3840}\alpha_{4,\frac{1}{2}}+\frac{1}{120}\alpha_{5,\frac{1}{2}}+\frac{4}{15}\alpha_{6,\frac{1}{2}}-\frac{81}{384}\beta_{\frac{5}{2},\frac{3}{2}}=-\frac{243}{3840}\\
C_{6,\frac{1}{2}} &= \frac{4}{45}r^6\alpha_{0,\frac{1}{2}}+\frac{1}{720}r^6\alpha_{1,\frac{1}{2}}+\frac{1}{46080}\alpha_{4,\frac{1}{2}}+\frac{1}{720}\alpha_{5,\frac{1}{2}}+\frac{4}{45}\alpha_{6,\frac{1}{2}}-\frac{243}{3840}\beta_{\frac{5}{2},\frac{3}{2}}=-\frac{720}{46080}
\end{aligned}\right\} \tag{14}$$

&

$$\left.\begin{aligned}
C_{0,2} &= \alpha_{0,1}+\alpha_{1,1}+\alpha_{2,1}+\alpha_{4,1}+\alpha_{5,1}+\alpha_{6,1}=-1\\
C_{1,2} &= -2r\alpha_{0,1}-r\alpha_{1,1}+\frac{1}{2}\alpha_{4,1}+\alpha_{5,1}+\frac{3}{2}\alpha_{6,1}-\beta_{3,2}=-2\\
C_{2,2} &= 2r^2\alpha_{0,1}+\frac{1}{2}r^2\alpha_{1,1}+\frac{1}{8}\alpha_{4,1}+\frac{1}{2}\alpha_{5,1}+\frac{9}{8}\alpha_{6,1}-2\beta_{3,2}=-2\\
C_{3,2} &= -\frac{4}{3}r^3\alpha_{0,1}-\frac{1}{6}r^3\alpha_{1,1}+\frac{1}{48}\alpha_{4,1}+\frac{1}{6}\alpha_{5,2}+\frac{27}{48}\alpha_{6,1}-2\beta_{3,2}=-\frac{4}{3}\\
C_{4,2} &= \frac{2}{3}r^4\alpha_{0,1}+\frac{1}{24}r^4\alpha_{1,1}+\frac{1}{384}\alpha_{4,1}+\frac{1}{24}\alpha_{5,1}+\frac{81}{384}\alpha_{6,1}-\frac{4}{3}\beta_{3,2}=-\frac{2}{3}\\
C_{5,2} &= -\frac{4}{15}r^5\alpha_{0,1}-\frac{1}{120}r^5\alpha_{1,1}+\frac{1}{3840}\alpha_{4,1}+\frac{1}{120}\alpha_{5,1}+\frac{243}{3840}\alpha_{6,1}-\frac{2}{3}\beta_{3,2}=-\frac{4}{15}\\
C_{6,2} &= \frac{4}{45}r^6\alpha_{0,1}+\frac{1}{720}r^6\alpha_{1,1}+\frac{1}{46080}\alpha_{4,1}+\frac{1}{720}\alpha_{5,1}+\frac{729}{46080}\alpha_{6,1}-\frac{4}{15}\beta_{3,2}=-\frac{4}{45}
\end{aligned}\right\} \tag{15}$$

Normalizing the coefficients $\alpha_{3,\frac{1}{2}}$, $\alpha_{4,1}$, $\alpha_{5,\frac{3}{2}}$ & $\alpha_{6,2}$ of $y_{n+\frac{1}{2}}$, y_{n+1}, $y_{n+\frac{3}{2}}$ & y_{n+2} respectively to 1. Solving equation (12), (13), (14) & (15) with the aids of Maple Software for the values of $\alpha_{j,i}$ and $\beta_{j,i}$ and Substituting the values in (8-11) gives the first, second, third & fourth point as

$$y_{n+\frac{1}{2}}=\frac{9(2r+1)}{16(4r+3)(r+1)r^2(40r^2-6r-7)}y_{n-2}+\frac{9(16r^2+8r+1)}{4r^2(40r^4+114r^3+55r^2-33r-14)(2r+3)}y_{n-1}-\frac{3(64r^4+9r^3+5r^2+12r+1)}{16r^2(40r^2-6r-7)}y_n+\frac{9(32r^2+32r^2+10r+1)}{4(r+1)(40r^2-6r-7)}y_{n+1}-\frac{3(64r^4+96r^3+52r^2+12r+1)}{(8r^2+18r+9)(40r^2-6r-7)}y_{n+\frac{3}{2}}+\frac{64r^4+96r^3+52r^2+12r+1}{16(40r^4+114r^3+55r^2-33r-14)}y_{n+2}-\frac{3(8r^2+6r+1)}{40r^2-6r-7}hf_{n+\frac{1}{2}} \tag{16}$$

$$y_{n+1} = \frac{1}{8(r+1)(4r+3)(4r+1)}y_{n-2} + \frac{1}{r^2(r^2+4r+4)(2r+3)}y_{n-1} - \frac{2r^2+3r+1}{24r^2}y_n + \frac{16(2r^2+3r+1)}{9(r+1)}y_{n+\frac{1}{2}} + \frac{16(2r^2+3r+1)}{3(8r^2+18r+9)}y_{n+\frac{3}{2}} - \frac{50r^3+193r^2+208r+62}{72(r^3+5r^2+8r+4)}y_{n+2} + \frac{2r+1}{12(r+2)}hf_{n+1} \qquad (17)$$

$$y_{n+\frac{3}{2}} = \frac{9(4r^2+12r+9)}{16(r+1)(r+1)(80r^3+292r^2+288r+81)(4r+1)r^2}y_{n-2} - \frac{9(16r^2+24r+9)}{4r^2(r+1)(r+2)(80r^3+292r^2+288r+81)}y_{n-1} + \frac{64r^4+288r^3+468r^2+324r+81}{16r^2(40r^2+126r+81)}y_n - \frac{3(64r^4+288r^3+468r^2+324r+81)}{(8r^2+16r+1)(40r^2+126r+81)}y_{n+\frac{1}{2}} + \frac{9(64r^4+288r^3+468r^2+324r+81)}{4(80r^4+372r^3+580r^2+369r+81)}y_{n+1} + \frac{3(64r^4+288r^3+468r^2+324r+81)}{16(r^2+3r+2)(40r^2+126r+81)}y_{n+2} + \frac{3(8r^2+18r+9)}{40r^2+126r+51}hf_{n+\frac{3}{2}} \qquad (18)$$

$$y_{n+2} = \frac{9(r^2+4r+4)}{(100r^3+411r^2+500r+186)(8r^2+6r+1)}y_{n-2} + \frac{72(r+1)}{r^2(50r^3+243r^2+376r+186)(2r+1)}y_{n-1} - \frac{3(r^4+6r^3+13r^2+12r+4)}{r^2(25r^2+84r+62)}y_n + \frac{128(r^4+6r^3+13r^2+12r+4)}{(8r^2+6r+1)(25r^2+84r+62)}y_{n+\frac{1}{2}} - \frac{72(r^3+5r^2+8r+4)}{(2r+1)(25r^2+84r+62)}y_{n+1} + \frac{384(r^4+6r^3+13r^2+12r+4)}{200r^4+1122r^3+2233r^2+1872r+62}y_{n+\frac{3}{2}} + \frac{6(r^2+3r+2)}{25r^2+84r+62}hf_{n+2} \qquad (19)$$

Substituting the variable step size ratio $r = \frac{1}{5}$ in (16), (17), (18) and (19), the stable method is represented in a tabular form

Table 2.1: Variable Step Size Ratio with the Stable Method obtained

Step size ratio (r)	Formulae
$r = \frac{1}{5}$	$y_{n+\frac{1}{2}} = -\frac{1}{244}y_{n-2} + \frac{5}{72}y_{n-1} - \frac{25}{16}y_n + \frac{25}{8}y_{n+1} - \frac{5}{7}y_{n+\frac{3}{2}} + \frac{25}{288}y_{n+2} - \frac{5}{3}hf_{n+\frac{1}{2}}$ $y_{n+1} = -\frac{1}{560}y_{n-2} + \frac{1}{45}y_{n-1} - \frac{1}{4}y_n + \frac{32}{45}y_{n+\frac{1}{2}} + \frac{32}{35}y_{n+\frac{3}{2}} - \frac{19}{48}y_{n+2} + \frac{1}{12}hf_{n+1}$ $y_{n+\frac{3}{2}} = \frac{17}{7904}y_{n-2} - \frac{49}{1976}y_{n-1} + \frac{1225}{3952}y_n - \frac{245}{247}y_{n+\frac{1}{2}} + \frac{3675}{1976}y_{n+1} - \frac{1225}{7904}y_{n+2} + \frac{104}{247}hf_{n+\frac{3}{2}}$ $y_{n+2} = -\frac{3}{665}y_{n-2} + \frac{16}{285}y_{n-1} - \frac{12}{19}y_n + \frac{512}{285}y_{n+\frac{1}{2}} - \frac{48}{19}y_{n+1} + \frac{1536}{665}y_{n+\frac{3}{2}} + \frac{14}{19}hf_{n+2}$ (20)

Analysis of the Method

In this section, order and Stability properties of the stable method (20) will be analysed.

1 Order of the Method

The order of the method (20) and its associated linear operator is given by

$$\backslash L[y(x);\ h] = \sum_{j=0}^{11}[C_j y(x+jh)] - h\sum_{j=0}^{11}[D_j y'(x+jh)] \qquad (21)$$

Where C_j , D_j are constant coefficients matrices and p is unique integer $s.t.$

$E_s = 0,\ s = 0,1, \ldots\ p$ and $E_{p+1} \neq 0$, where the E_s are constant Matrices

$for\ r = 1$, we have

$$E_0 = \sum_{J=0}^{11} C_J = 0$$

$$E_1 = \sum_{J=0}^{11}[jC_j - 2D_j] = 0$$

$$E_2 = \sum_{J=0}^{11}\left[\frac{1}{2!}j^2C_j - 2jD_j\right] = 0$$

$$E_3 = \sum_{J=0}^{11}\left[\frac{1}{3!}j^3C_j - 2\frac{1}{2!}j^2D_j\right] = 0$$

$$E_4 = \sum_{J=0}^{11}\left[\frac{1}{4!}j^4C_j - 2\frac{1}{3!}j^3D_j\right] = 0$$

$$E_5 = \sum_{J=0}^{11}\left[\frac{1}{5!}j^5C_j - 2\frac{1}{4!}j^4D_j\right] = 0$$

$$E_6 = \sum_{J=0}^{11}\left[\frac{1}{6!}j^6C_j - 2\frac{1}{5!}j^5D_j\right] = 0$$

$$E_7 = \sum_{j=0}^{11}\left[\frac{1}{7!}j^7C_j - 2\frac{1}{6!}j^6D_j\right] = \begin{bmatrix} -19280011/9072 \\ 122567/3150 \\ 3411415/1872 \\ -1204057/855 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (22)$$

Zero Stability Analysis of the Method

In this section, we investigate the zero and A- Stability property of the stable method.

1 Definition: According to [17], a linear multistep method is said to be zero stable if no root of the first characteristics polynomial has modulus greater than one and that any root with modulus one is simple.

2 Definition: According to [17], a linear multistep method is said to be an A-stable method if its stability region covers the entire negative half-plane.

To Obtained the zero stability of (20), the characteristic polynomial for $r = \frac{1}{5}$ can be obtain with the relation

For $r = \frac{1}{5}$

$$det(At^2 - Bt - C) = \frac{586800}{254881}t^8 - \frac{2283084}{1274405}t^7 - \frac{2489}{3823215}t^6 + \frac{1}{19116075}t^5 - \frac{11091911}{7646430}t^8h\lambda + \frac{38424}{115855}t^8(h\lambda)^2 - \frac{2887809}{2548810}t^7h\lambda + \frac{635055}{4078096}t^8(h\lambda)^3 + \frac{82674}{127405}t^2(h\lambda)^2 - \frac{945}{92684}t^8(h\lambda)^4 + \frac{5589}{370736}t^7(h\lambda)^3 + \frac{949}{1529286}t^6h\lambda + \frac{26}{1274405}t^6(h\lambda)^2 - \frac{1}{12744050}t^5h\lambda + \frac{21}{2039048}t^6(h\lambda)^3$$
(23)

Put $h\lambda = H = 0$ in (23)

We have

$$R_{\frac{1}{5}}(t,0) = \frac{586800}{254881}t^8 - \frac{2283084}{1274405}t^7 - \frac{2489}{3823215}t^6 + \frac{1}{19116075}t^5 \qquad (24)$$

Solving the Polynomials (24) for t using a maple. The following table is obtained for the roots of the polynomials.

Table 3.1: Zero Stability of the stable Method

Step size ratio (r)	Roots of the methods
$r = \frac{1}{5}$	$t = 0, 0, 0\ , 0\ , 0;\ -0.0003490211, ;\ 0.7871504297\ ;\ 1$

However, from the root of the method obtained, all root of the first characteristics polynomial has modulus less than or equal to one

A-Stability of the Method

This section focuses on obtaining stability regions for $r = \frac{1}{5}$ hybrid variable step size ratio. The stability region of the method (20) are determined through application of the standard linear test problem

$$y' = \lambda y, \quad \lambda < 0, \quad \lambda \text{ is complex.} \qquad (25)$$

The stability region is drawn in the $h\lambda$ plane and hence it takes all the values of $h\lambda$. The region which corresponds to the variable step size ratio $r = \frac{1}{5}$ method is drawn in a plot shown in fig. 1. The stability region for the method lies outside of the bounded region. Since most of the region in the left half plane is in the stability region, the method is almost A-stable (stiffly stable). Therefore, it can be concluded that the constructed hybrid variable step size block method is suitable for stiff problems because of its almost A-stable property.

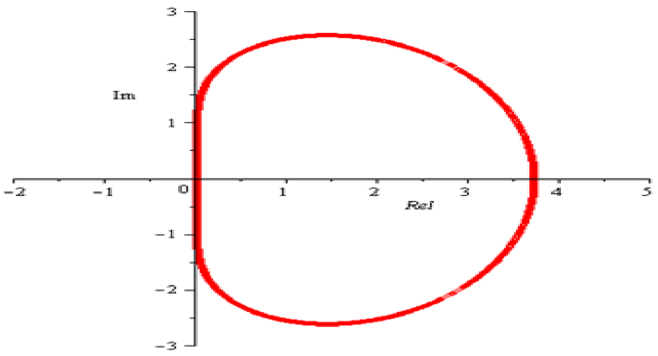

Figure 1: A- Stability Region of the Method when $r = \frac{1}{5}$

4. Test Problems

To validate the Performance of the method, below are some selected Stiff Nonlinear Initial Value Problem of ODEs to consider

Table 4.1: Sample of First Order Initial Value Problem of Stiff ODEs

S/n	Problems	Initial Conditions	Interval	Exact Solutions
1	$y' = -\frac{y^3}{2}$	$y(0) = 1$	$0 \le x \le 4$	$y(x) = \frac{1}{\sqrt{1+x}}$
2	$y' = 5e^{5x}(y-x)^2 + 1$	$y(0) = 0$	$0 \le x \le 1$	$y(x) = x - e^{-5x}$

Result and Discussions

Some chosen problems were solved using the developed method. The numerical result of the tested problems are put in tables to illustrate the performance of the block methods in solving first order nonlinear stiff ODEs. The acronyms below are used in the tables.

h = step-size;
MAX-ERR = Maximum Error;
NS= Number of Steps
IVPs = Initial Value Problems
OSHBM = Order and Stability of the reformulated hybrid block method for solving nonlinear ODEs.

Table 5.1: Numerical Result for the Test Problem 1

H	Method	Ns	Max-error
10^{-2}	OSHBM $(r = \frac{1}{5})$	100	8.23849 (-5)
10^{-3}	OSHBM $(r = \frac{1}{5})$	1000	8.36849 (-6)
10^{-4}	OSHBM $(r = \frac{1}{5})$	10000	8.43849 (-8)
10^{-5}	OSHBM $(r = \frac{1}{5})$	100000	8.63849 (-11)
10^{-6}	OSHBM $(r = \frac{1}{5})$	1000000	8.93849 (-12)

Table 5.1: Numerical Result for the Test Problem 2

H	Method	Ns	Max-error
10^{-2}	OSHBM $(r = \frac{1}{5})$	100	7.13551 (-5)
10^{-3}	OSHBM $(r = \frac{1}{5})$	1000	7.13551 (-6)
10^{-4}	OSHBM $(r = \frac{1}{5})$	10000	7.36551 (-7)
10^{-5}	OSHBM $(r = \frac{1}{5})$	100000	7.63551 (-8)
10^{-6}	OSHBM $(r = \frac{1}{5})$	1000000	7.83551 (-12)

From table 6.1 and 6.2 the newly reformulated hybrid variable step size block method is characterized by the decrease in error as the step length h tends to zero for the method at $r = \frac{1}{5}$. The accuracy also improves as the step length is reduced. Similarly, the solution at any fixed point improves as the step length h reduced. The maximum error indicates that the numerical result become closer to the exact solution as the step length tends to zero.

Conclusion

The Reformulated hybrid variable step size Block method for solving Nonlinear stiff Initial Value Problems of ODEs has been investigated. The analysis has shown that the method is zero and A-Stable capable of solving stiff initial value problem of nonlinear ordinary differential equation. The numerical result obtained is found to be accurate and more efficient as the certain step size (*h*) tends to zero.

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